EQUIVALENCE AND REDUCTION OF PAIRS OF HERMITIAN FORMS.*

BY MAYME IRWIN LOGSDON.

Introduction and Summary.—Of fundamental importance in the theory of matrices and forms has been the use by Weierstrass, Kronecker, and Frobenius of the theory of *elementary divisors* in the study of equivalence and reduction of pairs of bilinear and pairs of quadratic forms.

In this paper a generalization is made in that the basal theorems of the theory are extended to any hermitian λ-matrix, i.e., a matrix whose elements are polynomials of degree n in λ with coefficients in the field of complex numbers and are such that the conjugate of the element in the ith row and jth column is equal to the element in the jth row and ith column for $i, j = 1, 2, \cdots, n$.

Inasmuch as a linear substitution with matrix $\boldsymbol{P}$ on the variables of an hermitian form with matrix $\boldsymbol{a}$ gives a form with matrix $\boldsymbol{b} = \overline{\boldsymbol{P}}'\boldsymbol{a}\boldsymbol{P}$ where $\overline{\boldsymbol{P}}$ means the matrix formed from $\boldsymbol{P}$ by taking the conjugate imaginary of each element and $\boldsymbol{P}'$ means the transposed matrix $\boldsymbol{P}$, the extension of the general theory to the hermitian λ-matrix is justified by the proof in Part I of

THEOREM II. *If $b = \bar{p}'aq$ where p and q are non-singular and independent of λ, and where a and b are hermitian λ-matrices, then there exists a matrix P such that $b = \overline{P}'aP$.*

The special application is made to hermitian λ-matrices whose elements are linear in λ. Such will be the matrix of the pencil of forms

$$\lambda A - B = \sum_{i=1}^{n} \sum_{j=1}^{n} (\lambda a_{ij} - b_{ij})\bar{x}_i x_j.$$

The coincidence of the elementary divisors is found to be a necessary and sufficient condition for the equivalence of two pairs of hermitian matrices free of λ and for the equivalence of two pairs of hermitian forms.

In Part II, the Weierstrass reduction is shown to hold in case one of the forms is definite, a condition which insures reality of all the elementary divisors; in fact the Weierstrass method can be used for finding the contribution to the canonical form of any *real* elementary divisor. In the case however of conjugate complex elementary divisors, $(\lambda - \bar{a})^e$ and $(\lambda - a)^e$, it was found necessary and possible to regularize the λ-matrix with respect to the two conjugate imaginary linear factors simultaneously

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and also to expand the terms representing the adjoint form with respect to these two factors simultaneously.

In the actual work of reduction use was made of an algebraic simplification suggested and described by Dickson* in "Pairs of Bilinear or Quadratic Forms." The importance of this simplification is that in the case of bilinear forms the reduction may be accomplished rationally while in the case of quadratic or hermitian forms the computations are simplified if the constants c_κ which appear are held until the last step of the reduction before being absorbed in the variables.

The canonical form obtained † is given in Part I, Theorem V.

PART I—EQUIVALENCE.

We seek the conditions for equivalence of two pairs of hermitian forms:

$$\boldsymbol{A} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\bar{x}_i x_j, \qquad \boldsymbol{A}^* = \sum_{i=1}^{n} \sum_{j=1}^{n} a^*_{ij}\bar{x}_i x_j,$$

$$\boldsymbol{B} = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}\bar{x}_i x_j, \qquad \boldsymbol{B}^* = \sum_{i=1}^{n} \sum_{j=1}^{n} b^*_{ij}\bar{x}_i x_j,$$

with the respective matrices $\boldsymbol{a}$, $\boldsymbol{a}^*$, $\boldsymbol{b}$, $\boldsymbol{b}^*$ of which $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are non-singular, and where $\bar{a}_{ij} = a_{ji}$, etc., for $i, j = 1, \cdots, n$.

If a linear transformation with non-singular matrix, $\boldsymbol{c}$, whose elements are independent of λ,

$$x_i = \sum_{j=1}^{n} c_{ij} y_j \qquad (i = 1, \cdots, n),$$

with the induced transformation on the conjugate variables,

$$\bar{x}_i = \sum_{j=1}^{n} \bar{c}_{ij} \bar{y}_j \qquad (i = 1, \cdots, n),$$

be applied to $\boldsymbol{A}$ and $\boldsymbol{B}$, the transformed hermitian forms will have the respective matrices $\bar{\boldsymbol{c}}'\boldsymbol{ac}$ and $\bar{\boldsymbol{c}}'\boldsymbol{bc}$. If, now, these are to be the forms $\boldsymbol{A}^*$ and $\boldsymbol{B}^*$, we must have

$$\boldsymbol{a}^* = \bar{\boldsymbol{c}}'\boldsymbol{ac}, \qquad \boldsymbol{b}^* = \bar{\boldsymbol{c}}'\boldsymbol{bc}. \tag{1}$$

We shall show that a necessary and sufficient condition that equations (1) be satisfied is that the two hermitian λ-matrices, $\boldsymbol{m} = \boldsymbol{a} - \lambda\boldsymbol{b}$ and $\boldsymbol{m}^* = \boldsymbol{a}^* - \lambda\boldsymbol{b}^*$, have the same elementary divisors. We state the problem thus:

* Transactions A. M. Society, v. 10, 1909, p. 350.

† In the Proceedings of the London Mathematical Society, v. 32, 1900, pp. 321——, Bromwich obtains such a reduction by a special device, stating that "apparently this method (the Frobenius-Weierstrass method) cannot be extended so as to cover the analogous theory for conjugate imaginary substitutions, which would be applied to a pair of Hermite's forms."

Given any two hermitian λ-matrices, m and m^*, with elements polynomials in λ, to find necessary and sufficient conditions for the existence of a non-singular matrix c with elements independent of λ, such that $m^* = \bar{c}'mc$.

In proof we must show that if two hermitian λ-matrices are equivalent, the corresponding hermitian forms may be obtained, the one from the other, by a non-singular transformation on the variables. The first step of the proof will consist in establishing

THEOREM I. *If two hermitian λ-matrices, $m=a-\lambda b$ and $m^*=a^*-\lambda b^*$, are equivalent,* there exist two non-singular matrices, t and q, whose elements are independent of λ, such that*

$$m^* = tmq. \tag{2}$$

Proof: By the equivalence of m and m^* there exist non-singular λ-matrices t_0 and q_0 with determinants free of λ, such that

$$m^* = t_0mq_0. \tag{3}$$

Now divide t_0 by m^* and $(q_0)^{-1}$ by m † in such a way as to get matrices t_1, t, s_1, s which satisfy the relations

$$t_0 = m^*t_1 + t, \qquad (q_0)^{-1} = s_1m + s, \tag{4}$$

t and s being matrices whose elements do not involve λ. From (3) we get $t_0m = m^*q_0^{-1}$. Substituting here from (4) we have

$$m^*(t_1 - s_1)m = m^*s - tm. \tag{5}$$

Now the right member of (5) is a λ-matrix of at most the first degree, while for $t_1 - s_1 \neq 0$ the left member would be of at least the second degree. Hence $t_1 = s_1$ and

$$m^*s = tm. \tag{6}$$

Whence if we knew that s (and likewise t from (6)) were non-singular, we could write

$$m^* = tms^{-1} \tag{7}$$

and the theorem would be proved.

We proceed to show that s is non-singular. Substitute in the identity $I = q_0q_0^{-1}$ for q_0^{-1} from (4_2) and get

$$I = q_0s_1m + q_0s. \tag{8}$$

Now divide q_0 by m^* in such a way as to get

$$q_0 = q_1m^* + q, \tag{9}$$

* Two λ-matrices, m and m^*, are called equivalent (Bôcher, Introduction to Higher Algebra, p. 274) if there exist λ-matrices t_0 and q_0 each having as determinant a number not zero independent of λ such that $m^* = t_0mq_0$.

† Bôcher, p. 278.

where q is a matrix with elements free of λ. Substituting this value in (8) we have

$$I = q_0 s_1 m + q_1 m^* s + qs$$

which, by use of (6), may be written

$$(10) \qquad I - qs = (q_0 s_1 + q_1 t)m.$$

Since the left member does not contain λ we must have $q_0 s_1 + q_1 t$ identically zero, and therefore

$$(11) \qquad I = qs.$$

s is then non-singular, and we may write (6) in the form *

$$m^* = tmq.$$

Since $p = \bar{t}'$ is evidently a λ-matrix whose determinant equals the conjugate of the determinant t, we may express the definition of equivalence in the following form which is more convenient for hermitian forms:

Two hermitian λ-matrices, m and m^*, are *equivalent* if there exist non-singular matrices p and q with determinants free of λ, such that $m^* = \bar{p}'mq$.

The second step in solution of the original problem makes possible the extension of the theory of equivalence to the corresponding forms. It consists in proving

THEOREM II. *If* $b = \bar{p}'aq$, *where* p *and* q *are non-singular and independent of* λ, *and where* a *and* b *are hermitian* λ-*matrices, then there exists a matrix* P *such that*

$$b = \bar{P}'aP$$

and such that P *depends not on* a *or* b *but solely on* p *and* q.

We have by hypothesis

$$(1) \qquad b = \bar{p}'aq,$$

whence, since a and b are hermitian,

$$b = \bar{q}'ap.$$

Equating these two values of b we get

$$(2) \qquad \bar{q}'ap = \bar{p}'aq,$$

from which

$$(3) \qquad (\bar{q}')^{-1}\bar{p}'a = apq^{-1}.$$

If now we set $U = (\bar{q}')^{-1}\bar{p}'$, then $\bar{U}'$ will be pq^{-1} and (3) becomes

$$(4) \qquad Ua = a\bar{U}.$$

* This theorem holds if m and m^* have elements of degree p in λ.

From this we get at once

(5) $$U^2 a = a\bar{U}'^2;$$

and, in general,

(6) $$U^k a = a\bar{U}'^{(k)}.$$

From $a = a$ and (4), (5), (6) by using any set of arithmetical multipliers we get

(7) $$\chi(U)a = a\chi(\bar{U}'),$$

where $\chi(U)$ is any polynomial in U. Thus

(7′) $$a = [\chi(U)]^{-1}a\chi(\bar{U}').$$

Now we choose the polynomial $\chi(U) = V$ so that $V^2 = U$ and so that V is non-singular.* We have then from (7′)

(8) $$a = V^{-1}a\bar{V}'.$$

Substituting (8) in (1) we get

(9) $$b = p'V^{-1}a\bar{V}'q.$$

Now set $P = \bar{V}'q$, then $\bar{P}' = \bar{q}'V$ and $b = \bar{P}'aP$, as desired. For, from the definitions of U and V we have

$$U = V^2 = (\bar{q}')^{-1}\bar{p}'$$

from which we easily obtain $\bar{q}'V = \bar{p}'V^{-1}$.

These theorems permit the use of the general theory of λ-matrices. We state the theorems and definitions which are needed in the sequel:

I. If a and b are equivalent hermitian λ-matrices of rank r, and if $D_i(\lambda)$ is the greatest common divisor of the i-rowed determinants ($i \leqq r$) of a, then it is also the greatest common divisor of the i-rowed determinants of b.

II. Every hermitian λ-matrix of order n and rank r can be reduced by elementary transformations † to the normal form

$$\begin{Vmatrix} E_1(\lambda) & 0 & \cdots & 0 & \cdots & 0 \\ 0 & E_2(\lambda) & \cdots & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & E_r(\lambda) & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{Vmatrix},$$

where the coefficient of the highest power of λ in each of the polynomials $E_i(\lambda)$ is unity, and $E_i(\lambda)$ is a factor of $E_{i+1}(\lambda)$ for $i = 1, 2, \cdots, r-1$.

III. The greatest common divisor of the i-rowed determinants of an

* V is in fact a polynomial in U of degree less than n. See Bôcher, l.c., p. 299.

† Elementary transformations as defined in Bôcher, l.c., p. 262.

hermitian λ-matrix of rank r, when $i \leqq r$, is

$$D_i(\lambda) = E_1(\lambda)E_2(\lambda) \cdots E_i(\lambda),$$

where the E's are the polynomials of the last theorem.

IV. A necessary and sufficient condition that two hermitian λ-matrices of order n be equivalent is that they have the same rank r, and that for every value of i from 1 to r inclusive, the i-rowed determinants of one matrix have the same greatest common divisor as the i-rowed determinants of the other.

From the definition of the D's in III, we see that

$$E_i(\lambda) = \frac{D_i(\lambda)}{D_{i-1}(\lambda)}$$

$$(i = 1, 2, \cdots, r), \qquad (D_0(\lambda)) = 1).$$

Hence since the D's with the rank form a complete system of invariants, since the D's completely determine the E's as well as the elementary divisors, and since conversely the D's are completely determined by the E's or by the elementary divisors, we may state thus the

FUNDAMENTAL THEOREM: *A necessary and sufficient condition that two hermitian λ-matrices be equivalent is that they have the same rank and that the elementary divisors of one be identical respectively with the elementary divisors of the other.*

Definition: Two pairs of hermitian matrices $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a}^*$, $\boldsymbol{b}^*$ with elements free of λ will be called *equivalent* if there exist two non-singular matrices $\boldsymbol{p}$ and $\boldsymbol{q}$, also with elements not involving λ, such that

$$(1) \qquad \boldsymbol{a}^* = \bar{\boldsymbol{p}}'\boldsymbol{a}\boldsymbol{q}, \qquad \boldsymbol{b}^* = \bar{\boldsymbol{p}}'\boldsymbol{b}\boldsymbol{q}.$$

From this definition, Theorem I and the fundamental theorem we have

THEOREM III. *If $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a}^*$, $\boldsymbol{b}^*$ are two pairs of hermitian matrices independent of λ, and if $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are non-singular, a necessary and sufficient condition that these two pairs of matrices be equivalent is that the two λ-matrices*

$$\boldsymbol{m} = \boldsymbol{a} - \lambda\boldsymbol{b}, \qquad \boldsymbol{m}^* = \boldsymbol{a}^* - \lambda\boldsymbol{b}^*,$$

have the same elementary divisors.

Referring now to Theorem II, equivalence conditions for two pairs of matrices may be stated as follows:

THEOREM IV. *If $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{a}^*$, $\boldsymbol{b}^*$ are hermitian matrices independent of λ, and if $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are non-singular, a necessary and sufficient condition that a non-singular matrix $\boldsymbol{P}$ exist such that*

$$\boldsymbol{a}^* = \bar{\boldsymbol{P}}'\boldsymbol{a}\boldsymbol{P}, \qquad \boldsymbol{b}^* = \bar{\boldsymbol{P}}'\boldsymbol{b}\boldsymbol{P},$$

is that the matrices $a - \lambda b$ and $a^* - \lambda b^*$ have the same elementary divisors.

If in particular $b^* = b = I$, where I is the unit matrix, we have

$$I = \bar{P}'P$$

which defines an *orthogonal* hermitian matrix.

Corollary: If the characteristic matrices of a and a^* have the same elementary divisors there will exist an orthogonal matrix P such that

$$a^* = \bar{P}'aP \qquad \text{or} \qquad a^* = P^{-1}aP,$$

i.e., a^* is the transform of a by the orthogonal matrix P.

We have thus obtained the desired conditions for equivalence of two pairs of hermitian forms as stated in the first paragraph of Part I.

If the matrix of the form B is the unit matrix, referring to the last corollary we see that a transformation on the variables with orthogonal matrix P will transform the form B into B^* also with unit matrix. The λ-matrices

$$a - \lambda I, \qquad a^* - \lambda I$$

are now the characteristic matrices of the forms A and A^*, and as before a necessary and sufficient condition for the equivalence of the forms under orthogonal transformation is the coincidence of the elementary divisors of the characteristic matrices of the forms.

If B is a non-singular definite form, a preliminary transformation will transform it to the sum of hermitian squares, $\sum_1^n \bar{x}_i x_i$ with unit matrix, and since the roots of the determinant $|a - \lambda b| = 0$ are now the roots of the characteristic equation of a which are known to be always real,† we have the

Corollary: The elementary divisors of the pencil of hermitian forms $A - \lambda B$, where B is non-singular definite, are all real and of the first degree.‡

As in the quadratic case we have a further

*Corollary:** If A and B are hermitian forms and B is non-singular, a necessary and sufficient condition that it be possible to reduce A and B simultaneously by a non-singular transformation to forms into which only square terms (hermitian squares) enter is that all the elementary divisors of the pair of forms be of the first degree.

Finally, if both matrices are singular but at least one matrix of the pencil $\lambda_1 a + \lambda_2 b$ is non-singular, we may proceed as in the quadratic case § and obtain the desired canonical form.

The canonical form.

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* See Bôcher, l.c., p. 305, for the corresponding theorem for quadratic forms.

† G. Kowalewski-Einfuhrung in die Determinanten Theorie, p. 130.

‡ The proof in Bôcher, l.c., p. 170, for quadratic forms is applicable here.

§ Muth, l.c., p. 87.

THEOREM V. *If $c_1, c_2, \cdots, c_f$ are any real constants including zero, equal or unequal, if $a_g, a_h, \cdots, a_r$ are any complex numbers, equal or unequal, and if $e_1, e_2, \cdots, e_r$ are positive integers such that $e_1 + e_2 + \cdots + e_f + 2e_g + \cdots + 2e_r = n$, there exist pairs of hermitian forms in n variables, the first form being non-singular, which have the elementary divisors*

$$(\lambda - c_1)^{e_1},\ (\lambda - c_2)^{e_2},\ \cdots,\ (\lambda - c_f)^{e_f},\ (\lambda - \bar{a}_g)^{e_g},\ (\lambda - a_g)^{e_g},\ \cdots,\ (\lambda - \bar{a}_r)^{e_r},\ (\lambda - a_r)^{e_r}.$$

In proof after setting $e_1 + e_2 + \cdots + e_f = e$ we exhibit the forms

$$\begin{aligned} A = \sum_{i=1}^{e_1} \overline{X}_i X_{e_1-i+1} + \sum_{i=e_1+1}^{e_1+e_2} \overline{X}_i X_{2e_1+e_2-i+1} + \cdots \\ + \sum_{i=e-e_f+1}^{e} \overline{X}_i X_{2e-i+1} + \sum_{j=e+1}^{e+2e_g} \overline{X}_j X_{2e+2e_g-j+1} \\ + \cdots + \sum_{j=n-2e_r+1}^{n} \overline{X}_j X_{2n-2e_r-j+1}. \end{aligned}$$

$$\begin{aligned} B = \sum_{i=1}^{e_1} c_1 \overline{X}_i X_{e_1-i+1} + \sum_{i=e_1+1}^{e_1+e_2} c_2 \overline{X}_i X_{2e_1+e_2-i+1} + \cdots \\ + \sum_{i=e-e_f+1}^{e} c_f \overline{X}_i X_{2e-i+1} + \sum_{i=1}^{e_1-1} \overline{X}_i X_{e_1-i} + \sum_{i=e_1+1}^{e_1+e_2-1} \overline{X}_i X_{2e_1+e_2-i} + \cdots \\ + \sum_{j=e+1}^{e+e_g} a_g \overline{X}_j X_{2e+2e_g-j+1} + \sum_{j=e+e_g+1}^{e+2e_g} \bar{a}_g \overline{X}_j X_{2e+2e_g-j+1} + \cdots \\ + \sum_{j=n-2e_r+1}^{n-e_r} a_r \overline{X}_j X_{2n-2e_r-j+1} + \sum_{j=n-e_r+1}^{n} \bar{a}_r \overline{X}_j X_{2n-2e_r-j+1} \\ + \sum_{j=e+1}^{e+2e_g-1} \overline{X}_j X_{2e+2e_g-j} + \cdots + \sum_{j=n-2e_r+1}^{n-1} \overline{X}_j X_{2n-2e_r-j}. \end{aligned}$$

PART II—REDUCTION.

In the attempt to apply the methods of Weierstrass to reducing a pair of hermitian forms:

$$\boldsymbol{A} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \bar{x}_i x_j \qquad \text{and} \qquad \boldsymbol{B} = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} \bar{x}_i x_j,$$

where $\bar{a}_{ij} = a_{ji}$, $\bar{b}_{ij} = b_{ji}$, $|a_{ij}| \neq 0$, no trouble arises in finding the contribution to the canonical form due to any real linear factor of the λ-matrix, $\lambda \boldsymbol{a} - \boldsymbol{b}$, though proof is needed that certain theorems are actually extensible to this type of matrix. In dealing with complex linear factors however an essential modification is required. We shall first indicate the main steps in the process of reduction, then study in detail the separate cases where difference of treatment is required. Wherever the Weierstrass treatment as given in Muth's "Theorie und Anwendung der Elementartheiler" is valid without separation of the two cases, the details of the work will be omitted

and reference to Muth given. The following notations and definitions will be used:

S = determinant of the form $C = \lambda A - B$.

l_κ = exponent of the linear factor $(\lambda - r)$ in $D_\kappa(\lambda)$, i.e., in the greatest common divisor of all the κ-rowed minor determinants of S. There will be at least one κ-rowed minor determinant of S which contains $(\lambda - r)$ exactly l_κ times and is then defined to be *regular* with respect to this factor. We have

$$l_1 \leqq l_2 \leqq \cdots \leqq l_n$$

and also

$$l_i - l_{i-1} = e_i,$$

where $(\lambda - r)^{e_i}$ is an elementary divisor.

S_{ij} = cofactor of the element $\lambda a_{ij} - b_{ij}$ in S.

$S^{(\kappa)}$ = principal minor determinant with $n - \kappa$ rows obtained from S by deleting the first κ rows and the first κ columns. We note $S^{(\kappa)} = S_{\kappa\kappa}{}^{(\kappa-1)}$. For $\kappa = 0$, we define $S^{(0)} = S$, and for $\kappa = n$, $S^{(n)} = 1$.

S_ρ = the principal ρ-rowed minor determinant in the upper left-hand corner of the matrix.

$S_{\rho;\, ik}$ = $(\rho + 1)$-rowed minor determinant obtained by bordering S_ρ by the ith row and the kth column of the original matrix.

The main steps in the reduction are:

(1) Any hermitian λ-matrix may by elementary transformations * be regularized with respect to

(1) any real linear factor, $(\lambda - c)$.

(2) any pair of conjugate imaginary linear factors,

$$(\lambda - \bar{a}), \quad (\lambda - a).$$

* Elementary transformations of an hermitian λ-matrix are defined as follows:

1st. Interchanging two columns and the same two rows.

2d. Multiplying the elements of a column by a number m independent of λ and then multiplying the elements of the corresponding row by the conjugate of m.

3d. Adding to the elements of the kth column the products of the corresponding elements of the jth column $(j \neq k)$ by a polynomial, $\psi(\lambda)$, then adding to the elements of the kth row the products of the corresponding elements of the jth row each multiplied by the conjugate, $\bar{\psi}(\lambda)$, of the polynomial $\psi(\lambda)$.

The elementary transformations of the variables of the corresponding hermitian form which effect on the matrix of the form the above defined transformations are:

$$\begin{array}{ll} \text{1st.} & x_i = y_i \quad (i = 1, \cdots, n;\ i \neq j,\ i \neq k) \\ & x_j = y_k \\ & x_k = y_j. \\ \text{2d.} & x_i = y_i \quad (i = 1, \cdots, n;\ i \neq j) \\ & x_j = m y_j. \\ \text{3d.} & x_i = y_i \quad (i = 1, \cdots, n;\ i \neq j) \\ & x_j = y_j + \psi(\lambda) y_k. \end{array}$$

Definition: A matrix S is *regular* with respect to a real linear factor or with respect to a pair of conjugate imaginary linear factors if each of the principal minor determinants obtained from S by deleting the first k rows and columns, $(k = 1, 2, \cdots, n - 1)$, is so.

(2) The adjoint form may be written as a sum of terms in which every factor in a denominator is regular with respect to a given linear factor (or pair of factors); viz.,

$$(1) \qquad \sum_{i,j}^{1,n} \frac{S_{ij}}{S} \bar{u}_j u_i = \frac{\bar{X}'X'}{SS'} + \frac{\bar{X}''X''}{S'S''} + \cdots + \frac{\bar{X}^{(n)}X^{(n)}}{S^{(n-1)}S^{(n)}} .$$

(3) 1st. Each term on the right of (1) may be expanded with respect to a real linear factor, $(\lambda - c)$, the total coefficients of $1/(\lambda - c)^2$, $1/(\lambda - c)$ secured; finally, the total coefficients of $1/\lambda^2$, $1/\lambda$ secured. This will give the contribution of this particular linear factor, $\lambda - c$, to the canonical form.

2d. Each term on the right of (1) may be expanded with respect to $\lambda - \bar{a}$, $\lambda - a$ simultaneously and the total coefficients of $1/\lambda^2$, $1/\lambda$ secured.

(4) The adjoint form may be expanded by determinantal methods in descending powers of λ and the coefficients of $1/\lambda^2$, $1/\lambda$ obtained. These prove to be $\boldsymbol{B}$ and $\boldsymbol{A}$ respectively.

(5) A comparison of the results of (3) and (4) with Theorem V of Part I gives the desired expression for $\boldsymbol{A}$ and $\boldsymbol{B}$.

Proofs:

(I_1). Any hermitian λ-matrix, $\boldsymbol{S}$, may by elementary transformations be regularized with respect to a real linear factor, $\lambda - c$.

In proof we must show

(*a*) Every regular ρ-rowed minor determinant $(\rho > 1)$ contains at least one regular $(\rho - 1)$-rowed minor determinant as first minor.

(*b*) Every regular $(\rho - 1)$-rowed minor determinant $(\rho > 2)$ is contained in at least one regular ρ-rowed minor determinant as first minor.

(*c*) If a $(\rho - 1)$-rowed principal minor, $S_{\rho-1}$, is regular, but no ρ-rowed principal minor, $S_{\rho-1;\, i,\, i}$, containing it is regular, there is an elementary transformation of the variables which without disturbing the regularity of any $\boldsymbol{S}_k$ $(k = 1, \cdots, \rho - 1)$ will so transform the matrix $\boldsymbol{S}$ that there will be a regular ρ-rowed principal minor containing $\boldsymbol{S}_{\rho-1}$ as a first minor.

For proof of (*a*) and (*b*) see Muth, l.c., pp. 7–11.

Proof of (*c*): The existence of a regular ρ-rowed minor, $\boldsymbol{S}_{\rho-1;\, j,\, k}$, containing $\boldsymbol{S}_{\rho-1}$, is guaranteed by (*b*) above. Now apply to the variables of the form a preliminary transformation which will interchange the ρth and jth rows and columns and the $(\rho + 1)$st and kth rows and columns. We now have $\boldsymbol{S}_{\rho-1;\, \rho,\, \rho+1}$ for our regular ρ-rowed minor. Now apply the trans-

formation

$$T_m\colon x_i = y_i \qquad (i = 1, \cdots, n;\ i \neq \rho + 1),$$
$$x_{\rho+1} = y_{\rho+1} - m y_\rho.$$

The effect on the matrix is to subtract the products by m of the elements of the $(\rho + 1)$st column from the elements of the ρth column and then the products by $\bar{m}$ of the elements of the $(\rho + 1)$st row from the elements of the ρth row. Obviously S_k is not changed $(k = 1, \cdots, \rho - 1)$, but the principal minor, call it S'_ρ, of order ρ and containing $S_{\rho-1}$, is now regular. For we have

$$(2) \qquad S'_\rho = S_\rho - m S_{\rho-1;\,\rho,\,\rho+1} - \bar{m} S_{\rho-1;\,\rho+1,\,\rho} + \bar{m} m S_{\rho-1;\,\rho+1,\,\rho+1}.$$

Now S_ρ and $S_{\rho-1;\,\rho+1,\,\rho+1}$ each by hypothesis contains $\lambda - c$ more than l_ρ times; $S_{\rho-1;\,\rho,\,\rho+1}$ contains $\lambda - c$ exactly l_ρ times; $S_{\rho-1;\,\rho+1,\,\rho}$ is the conjugate of $S_{\rho-1;\,\rho,\,\rho+1}$ and therefore contains the real linear factor $\lambda - c$ exactly l_ρ times. It remains then to show that the sum $R = m S_{\rho-1;\,\rho,\,\rho+1} + \bar{m} \bar{S}_{\rho-1;\,\rho,\,\rho+1}$ does not have a higher power of $\lambda - c$ as factor than each part, for every m. Divide R by $(\lambda - c)^{l_\rho}$. $R = (\lambda - c)^{l_\rho}[mf(\lambda) + \overline{mf}(\lambda)]$, and, since $f(c)$ is not zero, if we set $m = 1/f(c)$ we have $mf(c) + \overline{mf}(c)$ not zero. Thus S'_ρ contains $\lambda - c$ exactly l_ρ times and is regular, as stated.

(I_2) Any hermitian λ-matrix, S, may by elementary transformations be regularized with respect to an imaginary linear factor. When this is done the matrix will be also regular with respect to the conjugate imaginary linear factor.

As before, (a) and (b) may be assumed for the factor $\lambda - a$ from the proof for bilinear forms. To prove (c) we apply the same transformation, T_m, and get as before the right member of equation (3). Remembering that the first and last terms are principal minors and hence have real coefficients, and that neither is regular with respect to $(\lambda - \bar{a})(\lambda - a)$, we may factor the right member of (3) thus:

$$[(\lambda - \bar{a})(\lambda - a)]^{l_\rho}[(\lambda - \bar{a})(\lambda - a)]^r f_1(\lambda) - m(\lambda - \bar{a})^p f_2(\lambda)$$
$$- \bar{m}\,(\lambda - a)^p \bar{f}_2(\lambda) + \bar{m} m[(\lambda - \bar{a})(\lambda - a)]^s f_3(\lambda),$$

where $r > 0$, $s > 0$, $p \geqq 0$, $f_2(a) \neq 0$.

If $p \neq 0$, the theorem is proved since $\lambda - a$ is a factor of all the terms in the brackets but one and consequently is not a factor of the sum. Thus S'_ρ is regular, as stated. If $p = 0$, we must show that $mf_2(\lambda) + \overline{mf}_2(\lambda)$ is not divisible by $\lambda - a$ where $f_2(a) \neq 0$ and $\bar{f}_2(\bar{a}) \neq 0$, i.e., we must show that m can be so chosen that $mf_2(a) + \overline{mf}(a) \neq 0$. This is the same condition reached before and is satisfied by $m = 1/f_2(a)$. Hence we may use the Jacobi transformation of the adjoint form, viz.,*

$$(3) \qquad \sum_{i,j}^{1,n} \frac{S_{ij}}{S}\,\bar{u}_j u_i = \frac{\bar{X}'X'}{SS'} + \frac{\bar{X}''X''}{S'S''} + \cdots + \frac{\bar{X}^{(n)}X^{(n)}}{S^{(n-1)}S^{(n)}},$$

* Muth, l.c., pp. 70–72.

with a determinant regular with respect to any linear factor, and where

$$
(4)\qquad
\begin{array}{rcl}
X' &=& S_{11}u_1 + S_{21}u_2 + S_{31}u_3 + \cdots + S_{n1}u_n \\
X'' &=& \qquad\quad S'_{22}u_2 + S'_{32}u_3 + \cdots + S'_{n2}u_n \\
X''' &=& \qquad\qquad\qquad S''_{33}u_3 + \cdots + S''_{n3}u_n \\
\cdot & \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
X^{(n)} &=& \qquad\qquad\qquad\qquad\qquad\qquad S_{nn}^{(n-1)}u_n
\end{array}
$$

(3) To decompose the general term on the right of (3) with respect to the real linear factor, $\lambda - c$, we may write

$$
(5)\qquad \frac{\overline{X}^{(\kappa)}X^{(\kappa)}}{S^{(\kappa-1)}S^{(\kappa)}} = \frac{\overline{X}^{(\kappa)}X^{(\kappa)}}{(\lambda - c)^{l_{\kappa-1}+l_\kappa}} \cdot \frac{1}{C_\kappa q^2},\text{*}
$$

where C_κ is the coefficient of the highest power of λ in $S^{(\kappa-1)}S^{(\kappa)}$.† It is evident that q^2, a polynomial in λ with real coefficients, will not contain $\lambda - c$ as a factor since $S^{(\kappa-1)}$ and $S^{(\kappa)}$ are regular, and we may then expand $X^{(\kappa)}/q$ and $\overline{X}^{(\kappa)}/q$ in power series in $\lambda - c$. Also, since in the definition of $X^{(\kappa)}$ in (4) the coefficient $S_{\kappa\kappa}^{(n-\kappa)}$ of u_κ is regular with respect to $\lambda - c$, $X^{(\kappa)}$, and consequently $\overline{X}^{(\kappa)}$ will contain $\lambda - c$ exactly l_κ times. Thus

$$
\frac{\overline{X}^{(\kappa)}}{q} = (\lambda - c)^{l_\kappa}[\overline{X}_{\kappa 1} + (\lambda - c)\overline{X}_{\kappa 2} + (\lambda - c)^2\overline{X}_{\kappa 3} + \cdots],
$$

$$
\frac{X^{(\kappa)}}{q} = (\lambda - c)^{l_\kappa}[X_{\kappa 1} + (\lambda - c)X_{\kappa 2} + (\lambda - c)^2 X_{\kappa 3} + \cdots].
$$

Hence

$$
\frac{\overline{X}^{(\kappa)}X^{(\kappa)}}{S^{(\kappa-1)}S^{(\kappa)}} = \frac{1}{C_\kappa}\frac{1}{(\lambda - c)^{e_\kappa}}[\overline{X}_{\kappa 1} + (\lambda - c)\overline{X}_{\kappa 2} + \cdots][X_{\kappa 1} + (\lambda - c)X_{\kappa 2} + \cdots],
$$

where the $X_{\kappa\mu}$ are linearly independent polynomials in c, u_κ, $\cdots$, u_n and the coefficients of the two forms $\boldsymbol{A}$ and $\boldsymbol{B}$ and are homogeneous in the u's. Thus dropping the subscript, κ, we may write the right member

$$
\frac{1}{C}\frac{1}{(\lambda - c)^e}[Z_1 + Z_2(\lambda - c) + Z_3(\lambda - c)^2 + \cdots]
$$

$$
= \cdots + \frac{Z_{e-1}}{C(\lambda - c)^2} + \frac{Z_e}{C(\lambda - c)} + \cdots.
$$

Now, defining $F_\kappa = Z_e = \sum_{i=1}^{i=e}\overline{X}_i X_{e-i+1}$ and $G_\kappa = Z_{e-1} = \sum_{i=1}^{i=e-1}\overline{X}_i X_{e-i}$ ($G_\kappa = 0$ for $e_\kappa = 1$) we have

$$
\frac{F_\kappa}{C_\kappa(\lambda - c)} + \frac{G_\kappa}{C_\kappa(\lambda - c)^2} = \frac{1}{C_\kappa}\left[\frac{F_\kappa}{\lambda} + \frac{cF_\kappa + G_\kappa}{\lambda^2} + \cdots\right].
$$

* For convenience in notation l_k shall henceforth represent the exponent of the factor $\lambda - c$ in the greatest common divisor of the $(n - \kappa)$-rowed minor determinants of S. We have then the inequalities, $l_0 \geqq l_1 \geqq l_2 \geqq \cdots \geqq l_n$, with $l_{\kappa-1} - l_\kappa = e_\kappa$.

† See Dickson, l.c.

There will be a similar expression obtained from each of the terms on the right of (3). The total contribution due to the real factor $\lambda - c$ to the canonical form is then obtained by taking the sum

$$\sum_{\kappa=1}^{f}\left[\frac{1}{C_\kappa}\frac{F_\kappa}{\lambda}+\frac{1}{C_\kappa}\frac{cF_\kappa+G_\kappa}{\lambda^2}\right],$$

where f is the number of distinct real linear factors. The numbers $1/C_\kappa$ may be now absorbed in the variables by $x' = x/\sqrt{C_\kappa}$ since the original forms and the transformations used have allowed irrationalities as well as imaginaries.

If now the above process of regularization and expansion be repeated for the remaining real linear factors, the total contribution of these factors is obtained.

To decompose with respect to $(\lambda - \bar{a})(\lambda - a)$ the general term of (3), we need to get the partial fractions of this term which have linear or quadratic denominators, viz., $(\lambda - \bar{a})$, $(\lambda - a)$, $(\lambda - \bar{a})^2$, $(\lambda - a)^2$, $(\lambda - \bar{a})(\lambda - a)$, since these and only these terms in the decomposition will contribute to the coefficient of $1/\lambda$ and $1/\lambda^2$. We set

$$\frac{\overline{X}^{(\kappa)}X^{(\kappa)}}{S^{(\kappa-1)}S^{(\kappa)}}=\frac{\overline{X}^{(\kappa)}X^{(\kappa)}}{C_\kappa[(\lambda-\bar{a})(\lambda-a)]^{l_{\kappa-1}+l_\kappa}q^2},$$

where q^2 is a polynomial in λ containing neither $\lambda - \bar{a}$ nor $\lambda - a$ as a factor and with unity for the coefficient of the highest power of λ. Now expand $X^{(\kappa)}/q$ in powers of $\lambda - a$ and $\overline{X}^{(\kappa)}/q$ in powers of $\lambda - \bar{a}$. Thus

$$\frac{X^{(\kappa)}}{q}=[(\lambda-\bar{a})(\lambda-a)]^{l_\kappa}[X_{\kappa 1}+(\lambda-a)X_{\kappa 2}+(\lambda-a)^2X_{\kappa 3}+\cdots],$$

$$\frac{\overline{X}^{(\kappa)}}{q}=[(\lambda-\bar{a})(\lambda-a)]^{l_\kappa}[\overline{X}_{\kappa 1}+(\lambda-\bar{a})\overline{X}_{\kappa 2}+(\lambda-\bar{a})^2\overline{X}_{\kappa 3}+\cdots],$$

where the $X_{\kappa\mu}$ are polynomials in $\bar{a}$, a, the coefficients of A and B, and homogeneous in $u_\kappa, \cdots, u_n$, while the $\overline{X}_{\kappa\mu}$ are the corresponding conjugate functions of $\bar{u}_\kappa, \cdots, \bar{u}_n$. We have then

$$\frac{\overline{X}^{(\kappa)}X^{(\kappa)}}{S^{(\kappa-1)}S^{(\kappa)}}=\frac{1}{C_\kappa}\frac{1}{(\lambda-\bar{a})^{e_\kappa}(\lambda-a)^{e_\kappa}}[\overline{X}_{\kappa 1}+(\lambda-\bar{a})\overline{X}_{\kappa 2}+(\lambda-\bar{a})^2\overline{X}_{\kappa 3}$$
$$+\cdots][X_{\kappa 1}+(\lambda-a)X_{\kappa 2}+(\lambda-a)^2X_{\kappa 3}+\cdots].$$

Omitting all κ subscripts, we may write the right member

$$\frac{1}{C}\left[\frac{\overline{X}_1}{(\lambda-\bar{a})^e}+\frac{\overline{X}_2}{(\lambda-\bar{a})^{e-1}}+\cdots+\frac{\overline{X}_e}{\lambda-\bar{a}}+\overline{X}_{e+1}+\overline{X}_{e+2}(\lambda-\bar{a})+\cdots\right.$$

$$+\bar{X}_{2e}(\lambda-\bar{a})^{e-1}+\cdots\Big]\Big[\frac{X_1}{(\lambda-a)^e}+\cdots+\frac{X_e}{\lambda-a}+X_{e+1}+\cdots$$

$$+X_{2e}(\lambda-a)^{e-1}+\cdots\Big]$$

$$=\frac{1}{C}\Big[\frac{\bar{X}_1X_{2e}(\lambda-a)^{e-1}}{(\lambda-\bar{a})^e}+\frac{\bar{X}_1X_{2e-1}(\lambda-a)^{e-2}}{(\lambda-\bar{a})^e}+\frac{\bar{X}_2X_{2e-1}(\lambda-a)^{e-2}}{(\lambda-\bar{a})^{e-1}}$$

$$+\frac{\bar{X}_2X_{2e-2}(\lambda-a)^{e-3}}{(\lambda-\bar{a})^{e-1}}+\cdots+\frac{\bar{X}_eX_{e+1}}{\lambda-\bar{a}}+\frac{\bar{X}_eX_e}{(\lambda-\bar{a})(\lambda-a)}$$

$$+\frac{\bar{X}_{e+1}X_e}{\lambda-a}+\cdots+\frac{\bar{X}_{2e}X_1(\lambda-\bar{a})^{e-1}}{(\lambda-a)^e}+\cdots\Big]$$

$$=\frac{1}{C}\frac{1}{\lambda}[\bar{X}_1X_{2e}+\bar{X}_2X_{2e-1}+\cdots+\bar{X}_eX_{e+1}+\cdots+\bar{X}_{2e}X_1]$$

$$+\frac{1}{C}\frac{1}{\lambda^2}[a(\bar{X}_1X_{2e}+\cdots+\bar{X}_eX_{e+1})+\bar{a}(\bar{X}_{e+1}X_e+\cdots+\bar{X}_{2e}X_1)$$

$$+\bar{X}_1X_{2e-1}+\bar{X}_2X_{2e-2}+\cdots+\bar{X}_{2e-1}X_1].$$

Thus we have found the part of the coefficient of $1/\lambda$ and $1/\lambda^2$ due to one term in the right member of (3) and contributed by the pair of linear factors $(\lambda-\bar{a})(\lambda-a)$ where the e in the last expression is the e_κ which belongs to the elementary divisors $(\lambda-\bar{a})^{e_\kappa}$, $(\lambda-a)^{e_\kappa}$. The total coefficient of $1/\lambda$ is seen to be (after the constant C_κ is absorbed in the variables)

$$\sum_{j=1}^{2e_k}\bar{X}_jX_{2e_k-j+1}$$

and the total coefficient of $1/\lambda^2$ is

$$\sum_{j=1}^{e_k}a\bar{X}_jX_{2e_k-j+1}+\sum_{j=e_k+1}^{2e_k}\bar{a}\bar{X}_jX_{2e_k-j+1}.$$

For other pairs of complex elementary divisors we proceed as here, then adding the coefficients of $1/\lambda$ obtained from all the linear factors, real and imaginary, and adding the coefficients of $1/\lambda^2$ obtained from all the linear factors, real and imaginary, we compare with the coefficients of these same powers of λ obtained by expanding the adjoint form

$$\sum_{i,j}^{1\cdots n}\frac{S_{ij}\bar{u}_ju_i}{S}$$

by determinanta methods,* thus obtaining the desired canonical forms given in Part I, Theorem V.

THE UNIVERSITY OF CHICAGO,
May, 1921.

* Muth, l.c., p. 81. $\sum_{i,j}^{1\cdots n}\frac{S_{ij}\bar{u}_ju_i}{S}=\frac{A}{\lambda}+\frac{B}{\lambda^2}+\cdots.$